

# Nonlinear three-wave interaction in pair plasmas

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It is shown that nonlinear three-wave interaction, described by vector-product-type nonlinearities, in pair plasmas implies much more restrictive conditions for a double energy transfer, as compared to electron-ion plasmas.

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An electron-ion (e-i) plasma, with a density gradient perpendicular to the ambient magnetic field vector, supports drift waves driven by the density gradient. In the nonlinear regime, the vector-product-type nonlinearity leads to a three-wave interaction which allows for both a direct and an inverse energy transfer [1–3], i.e., the transport of energy toward both shorter and longer wavelengths. A similar behavior is obtained also in rotating self-gravitating astrophysical clouds [4], where the inverse energy transfer implies the formation of large-scale structures on time scales much shorter than the gravitational contraction.

In the case of pair plasmas (electron-positron [5,6] or pair-ion plasmas), instead of the drift mode, one obtains convective cells [7]. In the past, the physics of electron-positron plasmas has been investigated mainly related to problems of active galactic nuclei and neutron stars. Nevertheless, experimental techniques have been developed for collecting and keeping positrons, and thus the antimatter plasma has been the subject of experimental investigations as well. However, the problem of particle annihilation that is inherent in electron-positron plasmas is absent in the recently experimentally produced pair-ion plasmas [8–11]. This experimental success has triggered an increased activity in the field in the past few years [12–17].

In the present work, we shall study the nonlinear wave interaction in pair plasmas. Hence, we assume a plasma with two components of equal mass and opposite charge (electron-positron or pair-ion), and we use the continuity and momentum equations for any of the two species in the form

$$\frac{\partial}{\partial t} \left( \frac{n_1}{n_0} \right) + \vec{\nabla}_\perp \cdot \vec{v}_{\perp 1} + \frac{\partial v_{z1}}{\partial z} + \vec{v}_{\perp 1} \cdot \vec{\nabla}_\perp n_0 + \frac{n_1}{n_0} \vec{\nabla} \cdot \vec{v}_1 + \vec{v}_1 \cdot \frac{\vec{\nabla} n_1}{n_0} = 0, \quad (1)$$

$$\left( \frac{\partial}{\partial t} + \vec{v}_1 \cdot \vec{\nabla} \right) \vec{v}_1 = \frac{q}{m} (-\vec{\nabla} \phi_1 + \vec{v}_1 \times \vec{B}_0). \quad (2)$$

The indices 0 and 1 here denote equilibrium and perturbed quantities, respectively. We study low-frequency perturbations  $\sim \exp(-i\omega t + i\vec{k}_\perp \cdot \vec{r} + ik_z z)$ ,  $|\partial/\partial t| \ll |\Omega| = |q|B_0/m$ , propagating nearly perpendicularly with respect to the magnetic

field vector  $\vec{B}_0 = B_0 \vec{e}_z$ , i.e.,  $|k_\perp| \gg |k_z|$ , and we allow for the presence of the equilibrium perpendicular density gradient  $\vec{\nabla}_\perp n_0$ . It will be shown that the density gradient effects play no role in the present problem.

From Eq. (2) we obtain approximately

$$\vec{v}_{\perp 1} = \frac{1}{B_0} \vec{e}_z \times \vec{\nabla}_\perp \phi_1 - \frac{1}{B_0 \Omega} \left( \frac{\partial}{\partial t} + \frac{1}{B_0} \vec{e}_z \times \vec{\nabla}_\perp \phi_1 \cdot \vec{\nabla}_\perp \right) \vec{\nabla}_\perp \phi_1, \quad (3)$$

$$\left( \frac{\partial}{\partial t} + \frac{1}{B_0} \vec{e}_z \times \vec{\nabla}_\perp \phi_1 \cdot \vec{\nabla}_\perp \right) v_z = -\frac{q}{m} \frac{\partial \phi_1}{\partial z}. \quad (4)$$

Using these two in Eq. (1), for any of the two species one may write

$$\begin{aligned} \mathcal{L} n_1 = & \frac{n_0}{B_0 \Omega} \mathcal{L} \nabla_\perp^2 \phi_1 - n_0 \frac{\partial v_{z1}}{\partial z} - \frac{n_0}{B_0} (\vec{e}_z \times \vec{\nabla}_\perp \phi_1) \cdot \vec{\nabla}_\perp \log n_0 \\ & - n_1 \vec{\nabla}_\perp \cdot \vec{v}_{\perp 1} - n_1 \frac{\partial v_{z1}}{\partial z} - v_{z1} \frac{\partial n_1}{\partial z}. \end{aligned} \quad (5)$$

Here,

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{B_0} \vec{e}_z \times \vec{\nabla}_\perp \phi_1 \cdot \vec{\nabla}_\perp.$$

Now, writing two Eqs. (5) for the two species and equating the corresponding expressions in view of the assumed quasineutrality, it is seen that the terms with the equilibrium density gradients exactly cancel each other out. The resulting combined equation for the species  $a, b$  consequently reads

$$\left( \frac{1}{B_0 \Omega_a} - \frac{1}{B_0 \Omega_b} \right) \mathcal{L} \nabla_\perp^2 \phi_1 + \frac{\partial}{\partial z} (v_{bz1} - v_{az1}) = 0. \quad (6)$$

Applying the operator  $\mathcal{L}$  onto Eq. (6) once more, and after using Eq. (4), one obtains

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + \frac{1}{B_0} \vec{e}_z \times \vec{\nabla}_\perp \phi_1 \cdot \vec{\nabla}_\perp \right) \\ & \times \left[ \left( \frac{\partial}{\partial t} + \frac{1}{B_0} \vec{e}_z \times \vec{\nabla}_\perp \phi_1 \cdot \vec{\nabla}_\perp \right) \nabla_\perp^2 \phi_1 \right] = -\Omega^2 \frac{\partial^2 \phi_1}{\partial z^2}. \end{aligned} \quad (7)$$

Linearizing Eq. (7) one then obtains the dispersion equation for the electrostatic convective cells as normal modes in this

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pair plasma, propagating almost perpendicularly to the magnetic field vector,

$$\omega^2 = \Omega^2 \frac{k_z^2}{k_\perp^2}, \quad \Omega = |q|B_0/m. \quad (8)$$

Note that using the Poisson equation instead of the quasineutrality will only modify the constant on the right-hand sides of Eqs. (7) and (8),  $\Omega^2 \rightarrow \Omega^2/[1 + \Omega^2/(2\omega_p^2)]$ ,  $\omega_p^2 = q^2 n_0/(\epsilon_0 m)$ . This modification is not essential for our work and will be neglected, implying that  $\Omega^2 \ll 2\omega_p^2$ .

In order to study the three-wave interaction, we use the nonlinear Eq. (7) assuming the perturbed potential (after omitting the previously used index 1) in the form

$$\phi(t) = \sum_{j=1}^{j=3} [\Phi_j(t) \exp(-i\omega_j t + i\vec{k}_j \cdot \vec{r}) + \Phi_j^*(t) \exp(i\omega_j t - i\vec{k}_j \cdot \vec{r})]. \quad (9)$$

Here, \* denotes the complex-conjugate quantity, and we shall use also

$$\frac{\partial}{\partial t} \Big|_j \rightarrow -i\omega_j + \frac{\delta}{\delta t}, \quad |i\omega_j| \gg |\delta/\delta t|,$$

where  $\delta/\delta t$  is the time derivative on a slow time scale, as a consequence of the nonlinear three-wave interaction. In this case, using Eq. (8), from Eq. (6) one obtains the following expression for time variation of the  $j$ th amplitude:

$$\begin{aligned} \frac{\delta \Phi_j(t)}{\delta t} = \frac{i}{2\omega_j k_j^2} & \left\{ \frac{1}{B_0} \left( \vec{e}_z \times \vec{\nabla}_\perp \frac{\partial \phi}{\partial t} \cdot \vec{\nabla}_\perp \right) \nabla_\perp^2 \phi \right. \\ & + \frac{2}{B_0} (\vec{e}_z \times \vec{\nabla}_\perp \phi \cdot \vec{\nabla}_\perp) \frac{\partial \nabla_\perp^2 \phi}{\partial t} + \frac{1}{B_0^2} (\vec{e}_z \times \vec{\nabla}_\perp \phi \cdot \vec{\nabla}_\perp) \\ & \left. \times [(\vec{e}_z \times \vec{\nabla}_\perp \phi \cdot \vec{\nabla}_\perp) \nabla_\perp^2 \phi] \right\}. \quad (10) \end{aligned}$$

On the right-hand side in Eq. (10),  $\phi$  should be taken as the summation (9). In view of the approximative calculation of the time-varying mode amplitude, the remaining time and space derivatives on the right-hand side imply  $\partial/\partial t, \vec{\nabla}_\perp$  implies  $\pm i\omega_l, \pm \vec{k}_l$  ( $l=1,2,3$ ), where  $\pm$  appears due to complex-conjugate expressions, and out of all terms introduced by the summation (9) one should keep only the resonant ones, corresponding to the  $-i\omega_j t + i\vec{k}_j \cdot \vec{r}$  on the left-hand side.

Without any loss of generality we further assume the following resonant conditions:

$$\omega_1 = \omega_2 + \omega_3, \quad \vec{k}_1 = \vec{k}_2 + \vec{k}_3. \quad (11)$$

The remaining task of calculating the nonlinear terms in Eq. (10) is presented below. For the mode  $\omega_1, \vec{k}_1$ , from the first term on the right-hand side in Eq. (10) we have

$$\begin{aligned} \Gamma_{1,1} & \equiv \frac{1}{B_0} \left( \vec{e}_z \times \vec{\nabla}_\perp \frac{\partial \phi}{\partial t} \cdot \vec{\nabla}_\perp \right) \nabla_\perp^2 \phi \\ & \Rightarrow \frac{i}{B_0} \vec{e}_z \cdot (\vec{k}_3 \times \vec{k}_2) (\omega_2 k_3^2 - \omega_3 k_2^2) \Phi_2 \Phi_3. \end{aligned}$$

Note that without the time derivative this nonlinear term would correspond to its counterpart in the electron-ion plasma [3].

Similarly, the second nonlinear term yields

$$\begin{aligned} \Gamma_{1,2} & \equiv \frac{2}{B_0} (\vec{e}_z \times \vec{\nabla}_\perp \phi \cdot \vec{\nabla}_\perp) \frac{\partial \nabla_\perp^2 \phi}{\partial t} \\ & \Rightarrow \frac{2i}{B_0} \vec{e}_z \cdot (\vec{k}_3 \times \vec{k}_2) (\omega_3 k_3^2 - \omega_2 k_2^2) \Phi_2 \Phi_3. \end{aligned}$$

Consequently, Eq. (10) for the mode  $\omega_1, \vec{k}_1$  becomes

$$\begin{aligned} \frac{\delta \Phi_1(t)}{\delta t} - Y_{13} & = - \frac{\vec{e}_z \cdot (\vec{k}_3 \times \vec{k}_2)}{2B_0 \omega_1 k_1^2} [\omega_2 k_3^2 - \omega_3 k_2^2 \\ & + 2(\omega_3 k_3^2 - \omega_2 k_2^2)] \Phi_2 \Phi_3. \quad (12) \end{aligned}$$

The meaning of the term  $Y_{13}$  is obvious from Eq. (10); it is the third nonlinear term there.

In the same manner from the time evolution (10) for the modes  $\omega_2, \vec{k}_2$  and  $\omega_3, \vec{k}_3$ , one obtains, respectively,

$$\begin{aligned} \frac{\delta \Phi_2(t)}{\delta t} - Y_{23} & = - \frac{\vec{e}_z \cdot (\vec{k}_1 \times \vec{k}_3)}{2B_0 \omega_2 k_2^2} [\omega_1 k_3^2 + \omega_3 k_1^2 \\ & - 2(\omega_3 k_3^2 + \omega_1 k_1^2)] \Phi_1 \Phi_3^*, \quad (13) \end{aligned}$$

$$\begin{aligned} \frac{\delta \Phi_3(t)}{\delta t} - Y_{33} & = - \frac{\vec{e}_z \cdot (\vec{k}_1 \times \vec{k}_2)}{2B_0 \omega_3 k_3^2} [\omega_1 k_2^2 + \omega_2 k_1^2 \\ & - 2(\omega_2 k_2^2 + \omega_1 k_1^2)] \Phi_1 \Phi_2^*. \quad (14) \end{aligned}$$

Equations (12)–(14) describe time evolution of the three modes due to their nonlinear interaction. For an arbitrary  $j$ th mode on the left-hand sides in Eqs. (12)–(14), the first and second nonlinear terms on the right-hand sides must include terms  $k$  and  $l$ , where  $k \neq l$  (including the combination with the complex-conjugate counterparts, too).

On the other hand, the third nonlinear term  $Y_{j3}$  includes products of all three amplitudes  $\Phi_j, \Phi_k, \Phi_l$  and, consequently, it cannot contain resonant exponential terms that would follow from the interaction between different modes. In other words, for the  $j$ th mode on the left-hand sides in Eqs. (12)–(14), the third nonlinear term  $Y_{j3}$  can only contain the self-interacting terms of the type  $c_j \cdot \Phi_j$ , where at the same time the interaction coefficient can only include terms of the form  $\Phi_k \Phi_k^*, \Phi_l \Phi_l^*$ , where  $k \neq j$  and  $l \neq j$ . Hence, although it describes the variation of the mode amplitude due to three-wave interaction, it does not directly contribute to a possible energy transfer toward larger and shorter wavelengths. It is a cubic nonlinearity yielding only a frequency shifts due to the modulational interaction. As such it in principle introduces a mismatch in the perfect resonant condition (11) for the frequencies, yet these effects will not be discussed here.

The double energy transfer can follow only from the first and second nonlinear terms in Eq. (10); therefore, only the contribution of those terms will be checked below. For that purpose, using the resonant conditions (11) we rewrite Eqs. (12)–(14) in a more symmetric form

$$\frac{\delta\Phi_1(t)}{\delta t} - Y_{13} = \frac{\vec{e}_z \cdot (\vec{k}_2 \times \vec{k}_3)}{2B_0\omega_1k_1^2} [\omega_2k_3^2 - \omega_3k_2^2 + 2(\omega_3k_3^2 - \omega_2k_2^2)]\Phi_2\Phi_3, \quad (15)$$

$$\frac{\delta\Phi_2(t)}{\delta t} - Y_{23} = \frac{\vec{e}_z \cdot (\vec{k}_2 \times \vec{k}_3)}{2B_0\omega_2k_2^2} [-\omega_1k_3^2 - \omega_3k_1^2 + 2(\omega_3k_3^2 + \omega_1k_1^2)]\Phi_1\Phi_3^*, \quad (16)$$

$$\frac{\delta\Phi_3(t)}{\delta t} - Y_{33} = \frac{\vec{e}_z \cdot (\vec{k}_2 \times \vec{k}_3)}{2B_0\omega_3k_3^2} [\omega_1k_2^2 + \omega_2k_1^2 - 2(\omega_2k_2^2 + \omega_1k_1^2)]\Phi_1\Phi_2^*. \quad (17)$$

It is interesting to compare these equations with drift-wave equations in electron-ion plasma [3], where the terms  $Y_{jk}$  are absent and the interaction coefficients for the three modes (without some unimportant common terms) are given, respectively, by

$$\alpha_1 = \vec{e}_z \cdot (\vec{k}_2 \times \vec{k}_3)(k_3^2 - k_2^2)\Phi_2\Phi_3, \quad (18)$$

$$\alpha_2 = \vec{e}_z \cdot (\vec{k}_2 \times \vec{k}_3)(k_1^2 - k_3^2)\Phi_1\Phi_3^*, \quad (19)$$

$$\alpha_3 = \vec{e}_z \cdot (\vec{k}_2 \times \vec{k}_3)(k_2^2 - k_1^2)\Phi_1\Phi_2^*. \quad (20)$$

The double energy transfer implies an energy transfer toward both shorter (the direct one) and longer (the inverse transfer) scales. Taking as an example

$$k_2^2 < k_3^2 < k_1^2, \quad (21)$$

for an electron-ion plasma, from Eqs. (18)–(20) it turns out that  $\alpha_3 < 0$ , implying the double energy transfer and the precipitation of energy from the intermediate mode  $k_3$  to the other two modes. In the opposite case  $k_2^2 > k_3^2 > k_1^2$ , we have  $\alpha_3 > 0$ ,  $\alpha_{1,2} < 0$  and the intermediate mode receives the energy from the two others. Similarly, for

$$k_3^2 > k_2^2 > k_1^2, \quad (22)$$

from Eqs. (18)–(20) it is seen that the intermediate mode  $k_2$  loses the energy,  $\alpha_2 < 0$ , and the double energy transfer takes place.

However, in our case dealing with pair plasmas, for positive frequencies and after using Eq. (11) to eliminate  $\omega_3$ , from Eqs. (15)–(17) (after disregarding the obvious positive and common terms) the signs of the coefficients of interaction are determined, respectively, by

$$\beta_1 = k_3^2(2\omega_1 - \omega_2) - k_2^2(\omega_1 + \omega_2), \quad (23)$$

$$\beta_2 = k_3^2(\omega_1 - 2\omega_2) + k_1^2(\omega_1 + \omega_2), \quad (24)$$

$$\beta_3 = k_2^2(\omega_1 - 2\omega_2) + k_1^2(\omega_2 - 2\omega_1). \quad (25)$$

As compared with the coefficients  $\alpha_j$  from an e-i plasma, the coefficients  $\beta_j$  appear to be more complex, with a dependence on frequencies, too.

For a more detailed comparison with the e-i plasma, we take the case (21) given above. In order to have the double energy transfer with the intermediate mode yielding the energy,  $\beta_3 < 0$ ,  $\beta_{1,2} > 0$ , from Eqs. (23)–(25) we obtain the following additional conditions for the frequencies and wave numbers:

$$\omega_1 > 2\omega_2, \quad k_1^2 > k_2^2 \frac{\omega_1 - 2\omega_2}{2\omega_1 - \omega_2}, \quad k_3^2 > k_2^2 \frac{\omega_1 + \omega_2}{2\omega_1 - \omega_2}. \quad (26)$$

Hence, the same double energy transfer in the pair-plasma imposes three more conditions. Yet, it is still possible and as an example we take the following set of numbers:  $\omega_1 = 3$ ,  $\omega_2 = 1$ ,  $\omega_3 = 2$  and  $k_1^2 = 3$ ,  $k_2^2 = 1$ ,  $k_3^2 = 2$ . These indeed satisfy all conditions (11), (21), and (26) and allow for the double energy transfer because  $\beta_1 = 6$ ,  $\beta_2 = 14$ ,  $\beta_3 = -14$ . From the derivations, and in particular from Eq. (10), it is clear that the  $k_j$  here imply the perpendicular components of the wave numbers. On the other hand, because both  $\Omega$  and the ratio  $k_z/k_\perp$  remain free (except for the assumption that the latter is much less than unity), the dispersion relation (8) will easily be satisfied, too.

Making one more comparison with the e-i plasma, we now use the condition (22) and analyze the coefficients (23)–(25). The requirement  $\beta_2 < 0$ , in view of Eq. (11), now yields

$$2\omega_2 > \omega_1 > \omega_2, \quad k_3^2 > k_1^2(\omega_1 + \omega_2)/(2\omega_2 - \omega_1). \quad (27)$$

The condition  $\beta_1 > 0$  yields  $k_3^2 > k_2^2(\omega_1 + \omega_2)/(2\omega_1 - \omega_2)$  and  $\omega_1 > \omega_2/2$ , with the latter being always satisfied in view of the condition (11). However, the obtained condition for the frequencies (27) in fact makes the third required condition  $\beta_3 > 0$  impossible. Hence, contrary to the e-i plasma case, which gives the double energy transfer, in the pair plasma for this case it is absent. In general, the main reason for these differences is the presence of frequencies in Eqs. (23)–(25), originating from the necessarily different decoupling (as compared to the e-i plasma) used in the derivations of Eq. (7).

To conclude, the pair plasma introduces a lot of interesting physical phenomena, with the nonlinear three-wave interaction discussed in this Brief Report being yet another example of it. We stress that pair properties imply that the results presented here are valid for both homogeneous and inhomogeneous environments.

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